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Diffraction of a plane shock by an analytic blunt body

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A study is made of the transient flow which results from the impingement of a plane shock on a blunt body. The analysis is based on Taylor-series expansions in the space and time variables of the flow properties and of the shape of the reflected shock. Coefficients of the series are determined numerically so as to satisfy the exact equations of motion and shock jump conditions. Convergence problems are ameliorated by recasting the series into continued fractions. While the analysis does not treat the transition from regular to Mach reflexion, it remains valid for all time in the subsonic region of the flow if the incident shock is sufficiently strong. In any case, it is accurate enough where it is valid to be useful for evaluation of more conventional numerical methods.

1. Introduction

The diffraction of a shock wave by a stationary body is a problem of some interest in connexion with the blast-wave loading of ground structures and with the starting of shock tubes and tunnels. However, most theoretical treatments are restricted to problems with conical symmetry (Pack 1964). Here our object is to predict the pressure distribution on smooth blunt bodies for as long as possible after the arrival of the incident shock. Thus, the only comparable work extant is Barnwell's (1967) recent numerical solution for the diffraction of a plane shock about a sphere.

Barnwell uses a finite-difference method similar to the well-known procedure of Lax (1954). Such methods account automatically for the presence of shocks and other discontinuities by (in effect) considering them to be regions in which the flow properties experience large but finite gradients and by applying the same difference equations within those regions as without. While the occurrence of Mach reflexion thus presents no particular computational difficulties to Barnwell, his results for the pressure on the body are naturally suspect in the initial stages of the diffraction, when the smeared-out reflected shock is still quite close to the body.

The present solution is also numerical, but treats the shocks as discontinuities. Results are obtained in the form of Taylor series in the space variables and time, the coefficients being determined from the exact three-dimensional conservation equations and boundary conditions. Using a computer, we obtain enough terms of the series to give us great confidence in the accuracy of our results, at least up to the time at which Mach reflexion occurs. Beyond this time our method furnishes only an incomplete picture of the flow field. If the incident shock is sufficiently strong, the solution is valid in principle for all time † near the stagnation point. However, for many practical purposes the present method can be used only as a supplement to more conventional numerical procedures like Barnwell's, in which case its role is to supply accurate data where they are unable to do so.

2. Determination of series solution

The problem we consider is the diffraction of a plane shock wave about a rigid stationary body, as illustrated in figure 1. The gas is assumed perfect. For simplicity we take the body to be either plane or rotationally symmetric about

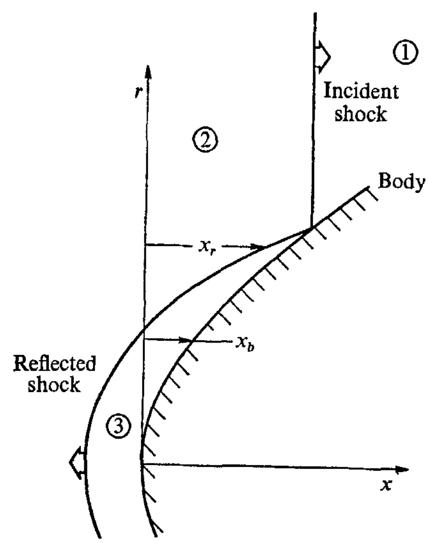

FIGURE 1. Co-ordinates and nomenclature. Numbers are used to identify flow regions defined by shock and body contours.

an axis normal to the incident shock. We also require that the body shape be analytic and so to have a power series expansion of the form

$$x_b = \sum_{i=1} x_b(i)\, r^{2i}. \tag{1}$$

The shape of the reflected shock is then assumed to have the expansion

$$x_r = \sum_{j=0} \sum_{k=0} x_r(j,k)\, r^{2j} t^k, \tag{2}$$

where the time t is measured from that at which the incident shock hits the body, while each flow property A (say) is expanded as follows:

$$A = \sum_{i=0} \sum_{j=0} \sum_{k=0} A(i,j,k)\,(x-x_r)^i r^{2j} t^k. \tag{3}$$

† In actuality, of course, the accuracy of the results deteriorates with increasing time.

Expansion in powers of $(x-x_r)$ rather than of x simplifies satisfaction of the shock jump conditions.

Given the properties of the undisturbed fluid (region 1 of figure 1) and the pressure ratio p_2/p_1 across the incident shock, we can easily find the speed V_i of that shock and the flow properties behind it (region 2) from the Rankine–Hugoniot shock jump conditions. With the body shape prescribed in the form (1), the problem is then to find the coefficients of the series (2) and (3) so as to satisfy the equations which govern the flow between the reflected shock and the body (region 3 of figure 1). These consist of the usual partial differential equations of inviscid flow, the jump conditions across the reflected shock, the condition of no flow through the body surface, and a geometric condition that the incident and reflected shocks meet the body at the same point, namely

$$V_i t = x_b = x_r \quad \text{at} \quad r = r_i(t) \quad \text{(say)}. \tag{4}$$

Since this condition is valid only while the reflexion is regular, our analysis breaks down when Mach reflexion occurs, to an extent which will be discussed in §4.

On substituting the various series into the governing equations and equating terms of like order in the independent variable, we obtain recursion formulas for the series coefficients. The details of deriving and of working with these formulas are very much the same as in the related shock-on-shock problem (Moran 1969). Suffice it to say here that the coefficients must be determined recursively according to their total order ($i+j$ for the coefficients $x_r(i,j)$, $i+j+k$ for $A(i,j,k)$). That is, we can find coefficients of total order N only after finding all those of total order $N-1$.

3. Utilization of series

Once the coefficients are available, we may attempt to use the series to calculate quantities of interest. Where the series converge, they may be used directly, with the accuracy of the results being estimated by examining various truncations of the series. However, as in related problems (Van Dyke 1958, Moran 1969), the regions in which series of interest converge are often limited by the occurrence of spurious singularities. For example, the series for the standoff distance of the reflected shock $x_r(0,t)$ clearly (by the ratio test or by Domb's (1965) method) contains an isolated singularity at a certain negative time t^*, so that the series diverges for $t > -t^*$.

Fortunately, methods have become available in recent years for the numerical construction of an analytic continuation of a power series beyond its basic region of convergence. Leavitt (1968) obtains excellent results for the inverse blunt-body problem by transforming the independent variable so as to obtain a more favourable location of the singularities of the series relative to the region of interest. An alternative procedure, which does not require any knowledge of the nature or location of the convergence-limiting singularities, is to recast the series into continued-fraction form. This method also works quite well in the blunt-body problem (Van Dyke 1966; Van Tuyl 1960, 1967; Moran 1969). In addition, it has been applied to the shock-on-shock problem (Moran 1969), and it will be used here as well.

The first step in our procedure is to convert the two- and three-dimensional series of interest into one-dimensional forms suitable for re-expansion as continued fractions. This we do by restricting our attention to curves $(x-x_r)/t =$ constant, $r^2/t =$ constant. Rewriting series like (2) and (3) in the forms

$$x_r = \sum_{k=0} \left[\sum_{j=0}^{k} x_r(j, k-j) \{r^2/t\}^j \right] t^k, \tag{5}$$

$$A = \sum_{k=0} \left[\sum_{i=0}^{k} \sum_{j=0}^{k-i} A(i, j, k-i-j) \{(x-x_r)/t\}^i \{r^2/t\}^j \right] t^k, \tag{6}$$

we fix $(x-x_r)/t$ and r^2/t and evaluate the polynomials in square brackets, thus obtaining power series in the single variable t. An advantage of this procedure over alternative approaches (see, e.g. Van Tuyl 1960, 1967) is that the coefficients of these series are not themselves infinite series, but only polynomials. Moreover, the coefficients of t^k in (5) and (6) are linear combinations of all those coefficients of the associated original series (2) or (3) of total order k. Since the coefficients of (2) and (3) are determined recursively according to their total order, it is easy to arrange the calculations so that the coefficients of t^k in (5) and (6) are known precisely; that is, within the limits of machine roundoff, but with no truncation error, up to the point at which those one-dimensional series are truncated.

Now, to any power series in one variable

$$f(t) = a_0 + a_1 t + a_2 t^2 + \ldots, \tag{7}$$

there corresponds a continued fraction

$$\cfrac{b_0}{1+\cfrac{b_1 t}{1+\cfrac{b_2 t}{1+\ldots}}} \tag{8}$$

whose coefficients b_n are determined uniquely in terms of $a_0, \ldots, a_n$ by the requirement that the power-series expansion of (8) about $t = 0$ agrees with the right side of (7). An efficient numerical procedure for finding the b_n's is the so-called quotient-difference algorithm (Henrici 1963).

It can be shown (Wall 1948, Shanks 1955) that, both when the function $f(t)$ is meromorphic and (at least in the special cases that have been examined in detail) when it has branch-point singularities, the continued-fraction expansion of f converges well beyond the circle of convergence of the corresponding power-series expansion. Indeed, in the case of interest, where t is real and positive, our experience seems to indicate that the continued-fraction expansion of f will converge almost everywhere up to the smallest real positive t at which f has a branch point.

However, convergence *per se* of the continued-fraction expansion of the solution is not enough for our purpose; we must have *rapid* convergence. We are required to truncate our series fairly early by, first of all, machine capacity. Using an IBM 360/65, we are able to compute only coefficients of total order $\leqslant 13$. Moreover, since the number of computations required to obtain a given coefficient is a rapidly increasing function of its total order, the higher-order coefficients are both expensive to obtain and subject to relatively large roundoff errors.

Some evidence of the convergence of the continued-fraction expansions is given in table 1, which compares results obtained from various truncations of the series solution for the standoff distance of the reflected shock at various times. As would be expected, the rate of convergence slows down with increasing time. However, since the power series used to form the relevant continued fraction was found to diverge at $t \sim 0{\cdot}31$, it can be seen that recasting the series has greatly extended its utility.

Time	Number of terms retained in series solution 11	12	13	14
0·2	−0·0618346	→	→	→
0·5	−0·1306775	→	→	→
1·0	−0·2091940	−0·2091956	−0·2091923	−0·2091906
2·0	−0·3003532	−0·3003761	−0·3002453	−0·3000487
4·0	−0·3836319	−0·3839854	−0·3829306	−0·3898388
8·0	−0·4415498	−0·4441937	−0·4374649	−0·4522165

TABLE 1. Results obtained from successive truncations of series solution for $x_r(0, t)$ in case 1 of table 2.

It should be emphasized that the time beyond which even a recast series becomes useless varies greatly from one series to the next. For an extreme example, the series for the flow properties in the immediate vicinity of the point at which the reflected shock meets the body become singular when the regular reflexion pattern depicted in figure 1 becomes impossible. Thus, the corresponding continued-fraction expansions also diverge beyond that time,† which is about 0·23 for the case to which the data of table 1 refer, or only a few per cent of the time for which the recast series yields reliable data on the standoff distance.

Because of this variability in the utility of the different series, their predictions were accepted only if three out of four successive truncations of the series gave results which were indistinguishable on a convenient graphical scale. In one instance this criterion resulted in the rejection of our highest-order results in favour of some of lower order. As can be seen in figure 5, using all 14 terms of the continued-fraction expansion of the standoff distance in one case (the same case dealt with in table 1) seemed to suggest the presence of a singularity in $x_r(0, t)$ at $t \sim 3{\cdot}0$. Of course, such behaviour is completely spurious. It is due

† Where these expansions do converge, however, they are in excellent agreement with results of a separate local analysis of the flow properties behind a regularly reflecting shock. Since the expansions involve, through the recursion formulas used to generate the Taylor-series solution, all the coefficients of all the series, this agreement gives us confidence in the correctness of our solution elsewhere as well.

only to roundoff errors accumulated in calculating the coefficients of the continued-fraction expansion of $x_r(0,t)$. Whether these errors simply introduce a negative coefficient into the continued fraction or whether they cause a misrepresentation of a removable singularity in the exact expansion is not known. However, reasonable results for $x_r(0,t)$ were obtained if its continued-fraction expansion was truncated at 13, 12 or 11 terms. Since those results were in substantial agreement among themselves (see also table 1), they were plotted in spite of the behaviour of the 'best' result.

4. Results

Calculations were made for the four cases described in table 2. In all cases the specific heat ratio was taken to be 1·4. Case 1 was selected for comparison with Barnwell's (1967) numerical results. The two cylinder cases were to have been compared with Bogoslavskii's (1966) experimental data; unfortunately, real-gas effects in the experiments were too strong to permit any meaningful comparisons.

Case	P_2/P_1	Body	M_2	t^*
1	35·77	Sphere	1·69994	0·23044
2	36·42	Circular cylinder	1·70300	0·23049
3	1·25	Sphere	0·15694	0·39451
4	2·485	Circular cylinder	0·60430	0·24575

TABLE 2. Identification of cases studied. M_2 = Mach number of the ultimate steady flow set up as time goes to infinity, while t^* = time at which regular reflexion becomes impossible, in units of the time required for the incident shock to move a length equal to the body radius.

Figures 2 and 3 give the pressure distributions on the body and on the axis of symmetry, respectively, at various times in the initial stages of the flow development for case 1. Similar results were obtained in the other cases. Time histories of the stagnation-point pressure and of the standoff distance of the reflected shock on the axis of symmetry are plotted for the various cases in figures 4 and 5. As noted previously, comparisons among the results obtained using various numbers of terms of the series solution lead us to believe that the truncation and roundoff errors of our results are negligible on the scales of these figures.

Also shown in figures 2–5 are some of Barnwell's (1967) numerical results for case 1. He uses finite-difference equations which take no particular notice of the presence of shocks in the flow field but which contain, in effect, viscous-like terms which smear out the discontinuities over several mesh widths. By changing the set of equations at alternate time steps, he is able to keep this smearing effect smaller than is usual, but he cannot entirely eliminate it. Thus, as is shown quite clearly in figures 2–4, his results for the pressure on the body are in considerable error for small times, but improve in accuracy as the smeared-out shock moves away from the body. Barnwell's results for the reflected-shock standoff distance, however, are in good agreement with ours even for short times.

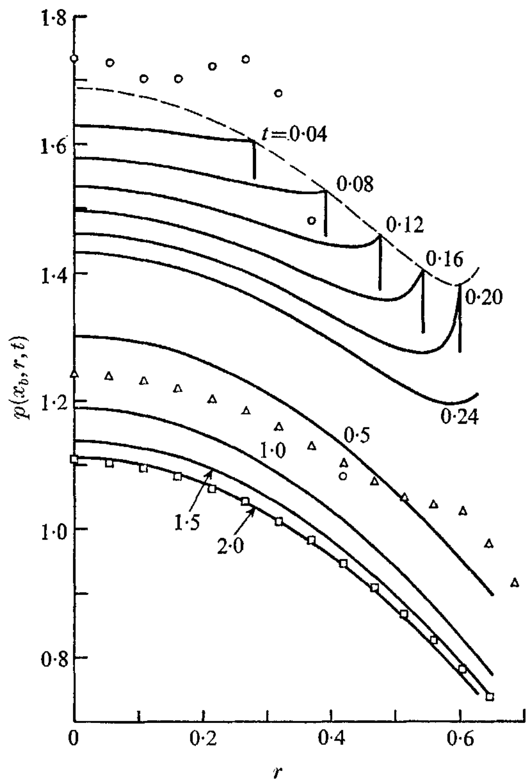

FIGURE 2. Pressure distribution on body at various times (case 1). Pressure made dimensionless with value of ρu^2 in region 2, distances with body radius R_B, times with time required for incident shock to wave distance equal to R_B. Barnwell's (1967) results: $\bigcirc$, t = 0·114; $\triangle$, t = 0·469; $\square$, t = 1·483.

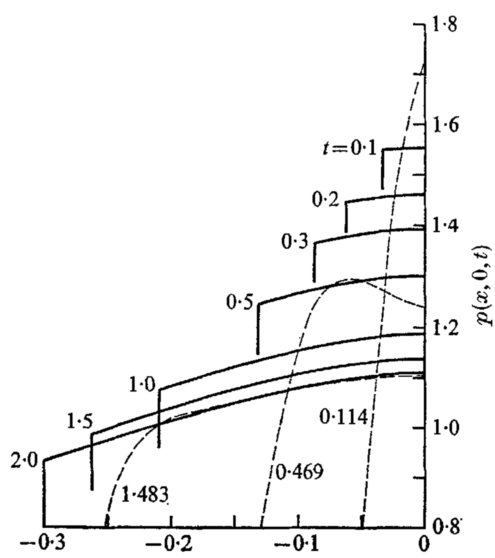

FIGURE 3. Pressure distribution on axis of symmetry at various times (case 1). Solid curves from present results, dashed from Barnwell (1967).

This is somewhat surprising, since the positioning of the shocks from smeared-out pressure distributions calls for a relatively subjective decision.

The main virtue of Barnwell's approach is that it is no worse after the occurrence of Mach reflexion than before, so that it yields an approximation to the

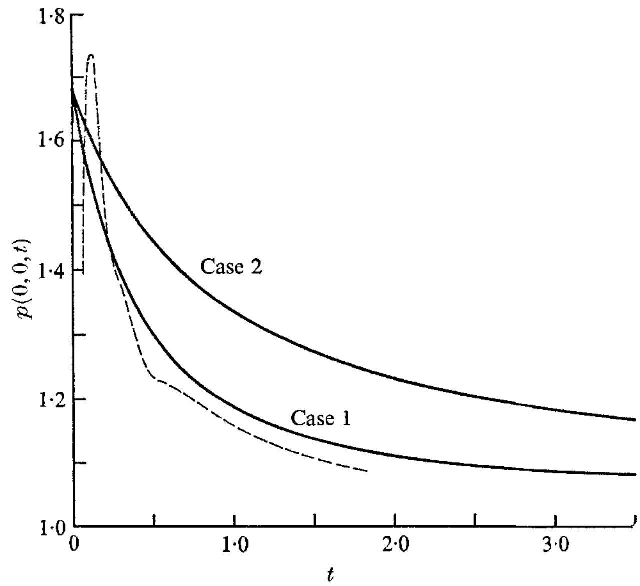

FIGURE 4. Stagnation-point pressure *vs.* time. Solid curves from present results, dashed from Barnwell's (1967) results for case 1.

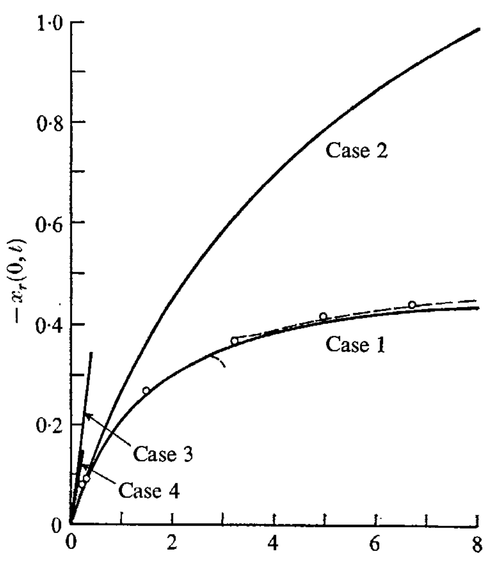

FIGURE 5. Standoff distance of reflected shock on axis of symmetry *vs.* time. Circles are Barnwell's (1967) data points for case 1. All other data are from present results. In case 1, dashed curve is from our 14-term continued-fraction, while the solid curve is from the 11-, 12- and 13-term continued-fractions.

complete history of the flow development, all the way to the final steady state. The present procedure, on the other hand, would have to be drastically modified (to incorporate a special analysis of the Mach reflexion process) in order to be completely valid beyond the time at which the assumed regular reflexion process becomes impossible. Since we are unable to describe the transition from regular to Mach reflexion analytically within the framework of inviscid flow theory, the best we can claim for our results is that they are valid at any point in the flow field so long as they are converging and until that point receives a signal that the transition has occurred.

		Number of terms retained in continued-fraction				
Case	Quantity	7	9	11	13	Correct result†
1	Stagnation point pressure	1·032	1·028	1·020	1·259	1·044
	Standoff distance	0·551	0·516	0·516	0·541	0·439
2	Stagnation point pressure	1·011	1·004	0·535	1·057	1·044
	Standoff distance	1·82	1·88	1·89	1·88	1·93

TABLE 3. Extrapolation of continued fractions to infinite time.

† Determined analytically for steady-state stagnation-point pressure and from correlation formulas of Ambrosio & Wortman (1962) for ultimate standoff distance.

In cases like 3 and 4, the flow velocities are always subsonic. Clearly every point in the flow field eventually learns of the occurrence of Mach reflexion in such cases. However, when the incident shock is sufficiently strong, as it is in cases 1 and 2, the flow behind the reflected shock can be shown to be supersonic relative to the body before regular reflexion becomes impossible. There are then large regions of the flow field, including the entire subsonic portion of the shock layer, which never know that Mach reflexion has occurred, and for which our solution is therefore valid so long as it is converging. Thus, while the data in the figures which refer to cases 3 and 4 have been (conservatively) terminated at the transition time, results are presented for cases 1 and 2 for times well beyond that at which Mach reflexion must take place, even though we are unable to describe the mechanics of the transition from regular reflexion.

Of course, as noted previously, there is still an upper bound on the time for which our series solution is useful, because of the limited number of terms of the series which are accurately known to us. Nevertheless, it is at least amusing to attempt to extrapolate our results to infinite time so as to predict some of the characteristics of the final steady-state flow. Observing that successive truncations of the continued fraction (8) alternately approach 0 and some finite value as $t \to \infty$, we can construct a sequence of approximants to the ultimate value of the associated function of time by considering only every other truncation. The results, displayed in table 3, indicate that roundoff errors prevent this from being a completely dependable procedure (note in particular the third entry in the third line), but that judicious examination of the results yields errors of only a few per cent in the steady-state stagnation pressure and in the final stand-off distance.

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